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ELASTIC CONTACT BETWEEN A FLEXIBLE CIRCULAR PLATE AND A TRANSVERSELY ISOTROPIC ELASTIC HALFSPACE

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Abstract—The axially symmetric flexural interaction of a uniformly loaded circular elastic plate resting in smooth contact with a transversely isotropic elastic halfspace is examined by using a variational method.

1. INTRODUCTION

The analysis of interaction between structural elements such as beams, plates, etc. and elastic media is of interest to several branches of engineering. Such solutions are of particular importance in analytical studies related to structural foundations resting on soil and rock media. This paper examines the application of an energy method to the analysis of the axisymmetric interaction between a uniformly loaded thin circular plate and a transversely isotropic elastic halfspace. The interface between the plate and the elastic halfspace is assumed to be smooth. Furthermore it is assumed that no separation occurs between the plate and the elastic medium.

The energy method of analysis of the interaction problem centers around the development of a total potential energy functional for the plate-elastic medium system, for prescribed plate deflection $w(r)$ which is indeterminate to within a set of arbitrary constants. The assumed form of $w(r)$ also satisfies the kinematic constraints of the axisymmetric deformation. The total potential energy functional consists of the strain energy of the halfspace region, the strain of the circular plate and the potential energy of the applied loads. The total potential energy functional thus developed is defined in terms of the undetermined constants characterizing the plate deflection. We may, however, eliminate two of these constants by invoking the Kirchhoff boundary conditions applicable to the free edge of the circular plate. The remaining constants are uniquely determined from the linearly independent algebraic equations generated from the minimization of the total potential energy functional.

The method of analysis outlined here is used to examine the flexural interaction of a uniformly loaded circular plate with a free edge resting in smooth contact with a transversely isotropic halfspace. The boundary plane of the transversely isotropic elastic halfspace is assumed to be perpendicular to the axis of elastic symmetry. The assumed deflected shape $w(r)$ is an even order polynomial in r up to the sixth order. This particular form of the deflected shape is assumed to represent, approximately, the flexural performance of circular plates of high relative rigidity. (The parameter, R_A, characterizing the relative rigidity of the plate-elastic medium system is defined by (21).)

The energy method of analysis yields analytical expressions for the central deflection, the differential deflection and central flexural moment in the thin circular plate. The accuracy of the energy estimate is compared with existing solutions for various choices of the relative rigidity parameter ($R_A \to 0$, or $R_A \to \infty$).

2. ANALYSIS

We consider the axisymmetric indentation of a transversely isotropic elastic halfspace by a thin flexible circular plate of thickness h and radius a. The plate is subjected to a uniform load of stress intensity p_0 over its entire surface (Fig. 1). Since no separation takes place at the interface, the deflected shape of the plate $w(r)$ also corresponds to the surface displacement of the halfspace in the z-direction, within the contact region $r \le a$.

For axially symmetric deformations of a thin plate, the flexural energy U_F is given by

$$U_F = \frac{D}{2}\int\int_S \left[\{\nabla^2 w(r)\}^2 - \frac{2(1-\nu_b)}{r}\frac{\mathrm{d}w(r)}{\mathrm{d}r}\frac{\mathrm{d}^2 w(r)}{\mathrm{d}r^2}\right] r\,\mathrm{d}r\,\mathrm{d}\theta \tag{1}$$

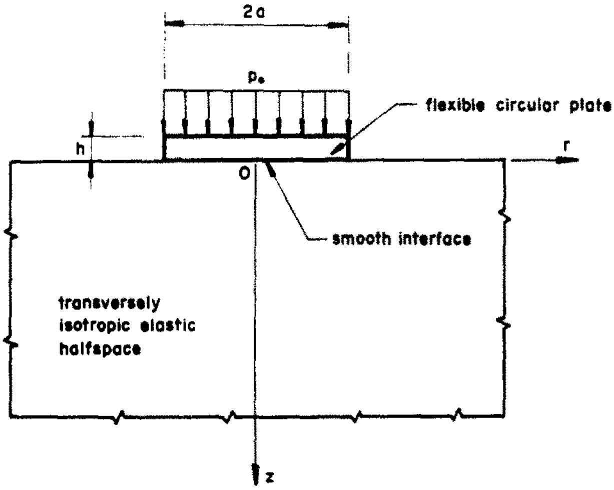

Fig. 1. Geometry of the interaction problem.

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}; \qquad D = \frac{E_b h^3}{12(1-\nu_b^2)} \tag{2}$$

and E_b and ν_b are respectively, the elastic modulus and Poisson's ratio for the plate material and S corresponds to the plate region.

The elastic strain energy in the transversely isotropic elastic halfspace can be developed by computing the work component of surface tractions which compose the interface contact stresses. Since the interface is assumed to be smooth, only the normal surface tractions contribute to the strain energy. The normal stresses generated as a result of the imposed surface displacement $w(r)$ can be uniquely determined from the integral equation methods developed by Elliott[1], Shield[2], England[3], Sveklo[4] and others for the analysis of mixed boundary value problems associated with transversely isotropic elastic materials. We consider the problem of a transversely isotropic elastic halfspace which is subjected to the axisymmetric displacement field

$$u_z = w(r) \qquad \text{for } z = 0, \qquad 0 < r < a \tag{3}$$

where u_z is the component of the displacement vector in the z-direction. The surface of the halfspace is subjected to the traction boundary conditions

$$\begin{aligned} \sigma_{zz} &= 0 \qquad \text{on } z = 0; \qquad a < r < \infty \\ \sigma_{rz} &= 0 \qquad \text{on } z = 0; \qquad 0 < r < \infty \end{aligned} \tag{4}$$

where σ_{zz} and σ_{rz} are the normal and shear stress components of the Cauchy stress tensor referred to the cylindrical polar coordinate system (r, θ, z). From the integral equation formulation of the mixed boundary value problem it can be shown that the compressive contact stress at the interface is given by

$$\sigma_{zz}(r, 0) = c_{44}(\gamma_1 - \gamma_2)\frac{1}{r}\frac{d}{dr}\int_r^a \frac{t g(t)\,dt}{\sqrt{(t^2 - r^2)}} \tag{5}$$

where

$$g(t) = \frac{2}{\pi}\left[\frac{k_1}{1+k_1} - \frac{k_2}{1+k_2}\right]^{-1}\frac{d}{dt}\int_0^t \frac{r w(r)\,dr}{\sqrt{(t^2 - r^2)}} \tag{6}$$

and γ_i, k_i ($i = 1, 2$) are constants related to the elastic constants c_{ij} of the transversely isotropic elastic material (see Appendix A). From the above results, the elastic strain energy of the transversely isotropic elastic halfspace due to the indentation $w(r)$ is given by

$$U_E = \frac{c_{44}(\gamma_1 - \gamma_2)(1+k_1)(1+k_2)a^3}{\pi(k_1 - k_2)} \int\int_S \frac{w(r)}{a}\left[\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{t}{\sqrt{(t^2 - r^2)}}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{r w(r)\,\mathrm{d}r}{a^2\sqrt{(t^2 - r^2)}}\right\}\mathrm{d}t\right]\mathrm{d}r\,\mathrm{d}\theta. \tag{7}$$

In general, the total potential energy of the externally applied axisymmetric load is given by

$$U_p = -\int\int_{S_p} p(r) w(r) r\,\mathrm{d}r\,\mathrm{d}\theta \tag{8}$$

where S_p is the region occupied by $p(r)$.

The total potential energy functional for the plate-elastic medium system (U) is obtained by the summation of (1), (7) and (8) (i.e. $U = U_F + U_E + U_P$). For the total potential energy functional to satisfy the principle of stationary potential energy we require

$$\delta U = 0 \tag{9}$$

where δU is the variation in the total potential energy. In order to apply the principle of total potential energy to the interaction problem we assume that the deflected shape $w(r)$ can be represented in the form

$$w(r) = a\sum_{i=0}^{n} C_{2i}\Phi_{2i}(r) \tag{10}$$

where C_{2i} are arbitrary constants and $\Phi_{2i}(r)$ are arbitrary functions which satisfy the kinematic requirements of the plate deformation. Of the $(n+1)$ arbitrary constants two can be eliminated by invoking the Kirchhoff boundary conditions[5] applicable for the free edge of the circular plate, i.e.

$$\begin{aligned} M_r(a) &= -D\left[\frac{\mathrm{d}^2 w(r)}{\mathrm{d}r^2} + \frac{\nu_b}{r}\frac{\mathrm{d}w(r)}{\mathrm{d}r}\right]_{r=a} = 0 \\ Q_r(a) &= D\left[\frac{\mathrm{d}}{\mathrm{d}r}\{\nabla^2 w(r)\}\right]_{r=a} = 0. \end{aligned} \tag{11}$$

Using the above conditions, the total potential energy functional for the plate-elastic medium system can be represented in terms of $(n-1)$ independent constants C_{2i}. The principle of total potential energy requires that U be an extremum with respect to the kinematically admissible deflection field characterized by C_{2i} (see, e.g. Sokolnikoff[6]). Hence

$$\frac{\partial U}{\partial C_{2i}} = 0 \qquad (i = 0, 1, \ldots, n-1). \tag{12}$$

The above minimization procedure yields $(n-1)$ linear equations for the undetermined constants C_{2i} ($i = 0, 1, 2, \ldots, n-1$).

3. ANALYSIS OF THE CIRCULAR PLATE PROBLEM

The formal theory developed in the preceding section is applied to the analysis of a uniformly loaded circular plate (i.e. $S_p = S$), with a free edge, resting on a transversely isotropic elastic halfspace. It is assumed that the deflected shape of the plate can be approximated by the power series

$$w(r) = a\sum_{i=0}^{3} C_{2i}\left(\frac{r}{a}\right)^{2i} \tag{13}$$

where C_{2i} are arbitrary constants. We note that in (13), the particular choice of functions corresponding to $\Phi_{2i}(r)$ give a kinematically admissible plate deflection and finite flexural moments and shearing force in the plate region $0 \leq r \leq a$. Upon satisfaction of the Kirchhoff free edge boundary conditions (11), the plate deflection (13) can be reduced to the form

$$w(r) = a\left[C_0 + C_2\left\{\left(\frac{r}{a}\right)^2 + \lambda_1\left(\frac{r}{a}\right)^4 + \lambda_2\left(\frac{r}{a}\right)^6\right\}\right] \tag{14}$$

where

$$\{\lambda_1; \lambda_2\} = \frac{(1+\nu_b)}{2(2+\nu_b)}\left\{-\frac{3}{2}; \frac{1}{3}\right\}. \tag{15}$$

The contact stress distribution corresponding to the imposed displacement field (14) can be determined by making use of the relationships (5) and (6); we have

$$\sigma_{zz}(r,0) = \frac{2c_{44}a}{\pi\Psi\sqrt{(a^2-r^2)}}\left[C_0 + C_2\left\{\sum_{i=0}^{3}\eta_{2i}\left(\frac{r}{a}\right)^{2i}\right\}\right], \tag{16}$$

where

$$\Psi = \frac{(k_1-k_2)}{(\gamma_1-\gamma_2)(1+k_1)(1+k_2)} \tag{17}$$

and η_{2i} are constants defined in Appendix A. Also, by making use of $w(r)$ as defined by (14), the total potential energy functional U reduces to the form

$$U = \frac{2c_{44}a^3}{\Psi}[C_0^2 + C_0C_2\chi_1 + C_2^2\chi_2] + \pi DC_2^2\chi_3 - \pi p_0a^3[C_0 + \chi_4C_2] \tag{18}$$

and the constants χ_n $(n = 1, 2, 3, 4)$ are also defined in Appendix A. The constants C_0 and C_2 can be determined from the equations which are generated from the minimization conditions

$$\frac{\partial U}{\partial C_0} = 0; \qquad \frac{\partial U}{\partial C_2} = 0. \tag{19}$$

The deflected shape of the uniformly loaded circular foundation corresponding to (13) is given by

$$w(r) = \frac{\pi a p_0\Psi}{2c_{44}\{\chi_1^2 - 4\chi_2 - 2R_A\chi_3\}}\left[\chi_1\chi_4 - 2\chi_2 - R_A\chi_3 + \{\chi_1 - 2\chi_4\}\left\{\left(\frac{r}{a}\right)^2 + \lambda_1\left(\frac{r}{a}\right)^4 + \lambda_2\left(\frac{r}{a}\right)^6\right\}\right] \tag{20}$$

and R_A is a relative rigidity parameter of the circular plate-transversely isotropic elastic halfspace system defined by

$$R_A = \frac{\pi}{12}\frac{\Psi}{(1-\nu_b^2)}\frac{E_b}{c_{44}}\left(\frac{h}{a}\right)^3. \tag{21}$$

The accuracy of the solution for the plate deflection[(20)] developed by the variational procedure can be examined by assigning suitable limits to the relative rigidity parameter R_A.

4. LIMITING CASES

(i) *Infinitely rigid plate*

As the relative rigidity parameter $R_A \to \infty$, the interaction problem reduces to that of the smooth indentation of a transversely isotropic elastic halfspace by a rigid circular punch. In the limit as $R_A \to \infty$, (20) gives the following expression for the constant displacement w_0

$$w_0 = \frac{\pi a p_0 (k_1 - k_2)}{4 c_{44}(1 + k_1)(1 + k_2)(\gamma_1 - \gamma_2)}. \tag{22}$$

The above result is in agreement with the expressions obtained by Elliott[1] and Shield[2] for the rigid displacement of a circular punch on a transversely isotropic elastic halfspace which were obtained by considering, separately, integral equation methods and results of potential theory respectively.

(ii) *Flexible circular loading*

As $R_A \to 0$, the interaction problem reduces to that of the axisymmetric loading of a transversely isotropic elastic halfspace by a uniform circular load of radius a and stress intensity p_0. The two results of particular engineering interest are the maximum deflection $w(0)$ and the differential deflection $\{w(0) - w(a)\}$ within the uniformly loaded area.

In the limit as $R_A \to 0$, the expression (20) yields

$$\{w(0)\}_{\text{Energy}} = \frac{p_0 a (k_1 - k_2)}{c_{44}(1 + k_1)(1 + k_2)(\gamma_1 - \gamma_2)}\{1.045\}. \tag{23a}$$

The exact solution for the central deflection of a transversely isotropic elastic halfspace subjected to a uniform circular load can be generated by making use of the results given by Elliott[1] and Shield[2] for the surface and interior loading of a halfspace. We obtain

$$\{w(0)\}_{\text{Exact}} = \frac{p_0 a (k_1 - k_2)}{c_{44}(1 + k_1)(1 + k_2)(\gamma_1 - \gamma_2)}. \tag{23b}$$

The energy estimate for the central deflection of a uniformly loaded area overpredicts the exact solution by approximately 4.5%.

Similarly, the energy estimate for the differential deflection of the uniformly loaded region is

$$\{w(0) - w(a)\}_{\text{Energy}} = \frac{p_0 a (k_1 - k_2)}{c_{44}(1 + k_1)(1 + k_2)(\gamma_1 - \gamma_2)}\{0.364\}. \tag{24a}$$

The corresponding exact solution for the differential deflection is

$$\{w(0) - w(a)\}_{\text{Exact}} = \frac{p_0 a (k_1 - k_2)}{c_{44}(1 + k_1)(1 + k_2)(\gamma_1 - \gamma_2)}\left\{\frac{\pi - 2}{\pi}\right\}. \tag{24b}$$

The energy estimate for the differential deflection overpredicts the exact solution by approximately 0.3%.

It may also be verified that in the limiting case of an isotropic material, the constants c_{ij} can be related to the Lame constants λ and μ as follows: $c_{11} = c_{33} = \lambda + 2\mu$, $c_{12} = c_{13} = \lambda$, $c_{44} = \mu$. If we let $\nu_1, \nu_2 \to 1$, we find that $\Psi \to (1 - \nu)$ and the results (23) and (24) reduce to their counterparts for isotropic elastic materials.

5. FLEXURAL MOMENTS

The flexural moments in the uniformly loaded plate can be determined by making use of the expression for the plate deflection (20) and the relationships

$$\begin{bmatrix} M_r(r) \\ \\ M_\theta(r) \end{bmatrix} = -D \begin{bmatrix} \dfrac{d^2 w}{dr^2} & \dfrac{1}{r}\dfrac{dw}{dr} \\ \dfrac{1}{r}\dfrac{dw}{dr} & \dfrac{d^2 w}{dr^2} \end{bmatrix} \begin{bmatrix} 1 \\ \\ \nu_b \end{bmatrix}. \tag{25}$$

In general any inaccuracies in $w(r)$ as computed by the energy method are magnified in the computation of M_r and M_θ owing to the presence of derivatives up to the second order. A more accurate estimate of the flexural moments induced in the plate can be obtained by computing the flexural moments induced in the plate due to the combined action of the external uniform load p_0 and the contact stress distribution $\sigma_{zz}(r)$. Using this technique, it can be shown that the central flexural moment (M_0) in the circular plate is given by

$$M_0 = \frac{p_0 a^2(3+\nu_b)}{16} - \int_0^a \frac{\zeta}{2}\sigma_{zz}(\zeta, 0)\left\{\frac{(1-\nu_b)}{2}(a^2-\zeta^2) - (1+\nu_b)\ln\left(\frac{\zeta}{a}\right)\right\} d\zeta. \tag{26}$$

Using (16) in (25) we obtain

$$M_0 = p_0 a^2\left\{\frac{(3+\nu_b)}{16} - \frac{(1-\nu_b)}{4} m_0 + \frac{(1+\nu_b)}{2} m_2\right\} \tag{27}$$

where m_0 and m_2 are defined in Appendix A.

6. NUMERICAL RESULTS AND DISCUSSION

The assumption of tensionless contact at the frictionless interface is central to the developments presented in the preceding sections. For the energy solution to be physically admissible it is necessary that the contact stresses developed at the interface remain compressive for various combinations of the relative rigidity parameter R_A and the elastic constants c_{ij}. Should the contact stresses become tensile in any region of the interface then the interaction problem becomes one of unbonded or unilateral contact between the thin plate and the transversely isotropic elastic medium. Accounts of such investigations are given by Weitsman[7], Gladwell and Iyer[8], Gladwell[9], de Pater and Kalker[10] and Selvadurai[11]. Frictionless contact between plates and elastic media, induced by highly localized or concen-

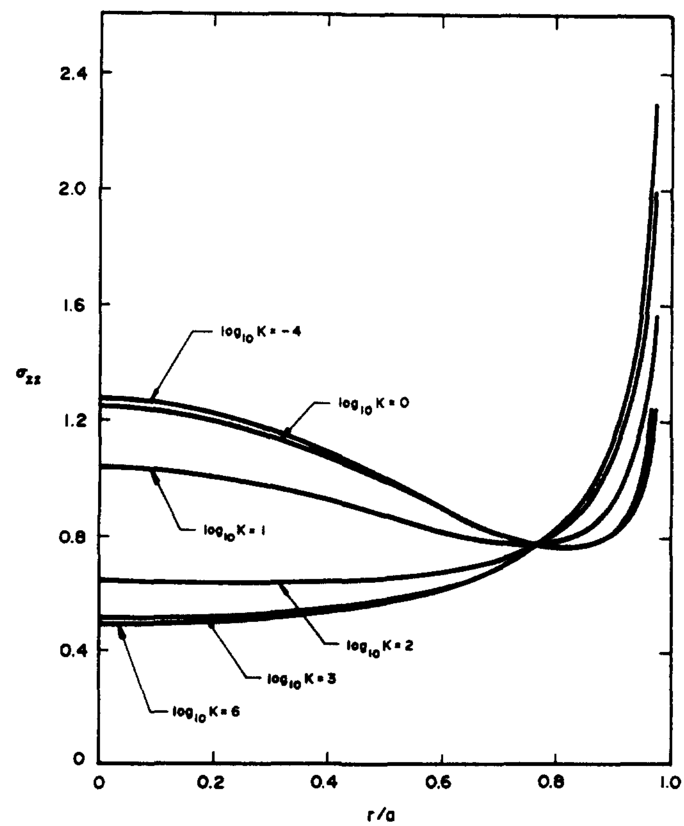

Fig. 2. Contact stress distribution at the interface: halfspace material, isotropic.

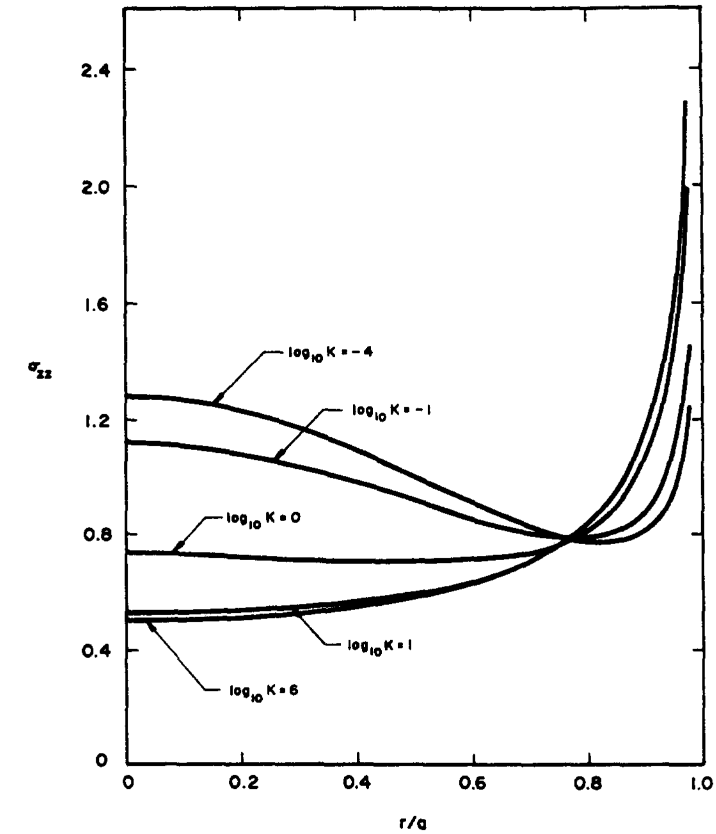

Fig. 3. Contact stress distribution at the interface: halfspace material, magnesium.

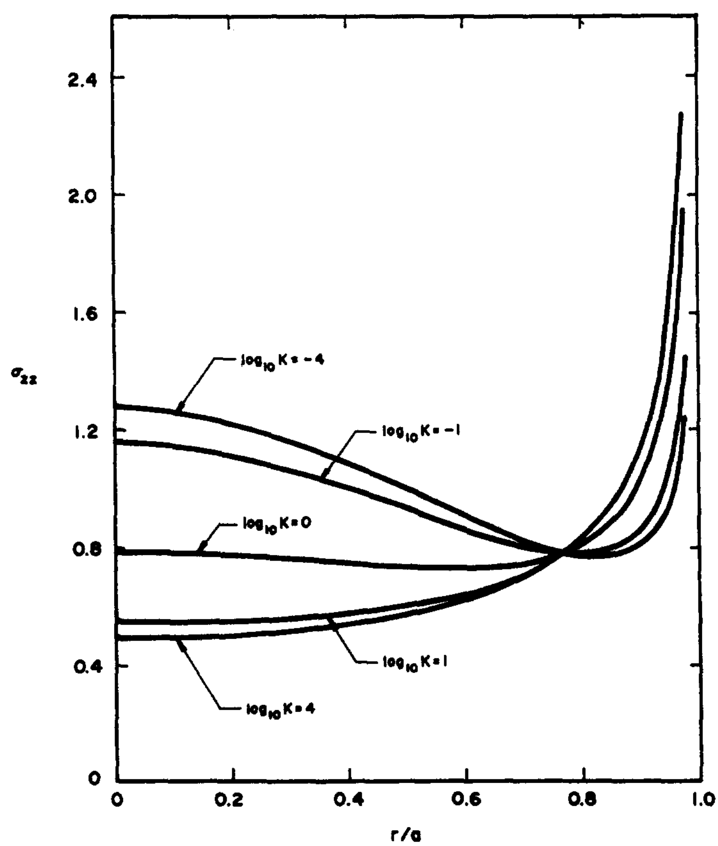

Fig. 4. Contact stress distribution at the interface: halfspace material, cadmium.

trated loads are susceptible to such separation effects. In the present study, some indication of the nature of the contact stress distribution can be obtained by assigning limiting values for the relative rigidity parameter R_A (i.e. $R_A \to 0$ and $R_A \to \infty$). In the former case, the applied stress p_0 is directly transmitted to the interface without any flexural interaction. Therefore, the contact stresses are always compressive for all choices of the material parameters c_{ij}. The energy estimate for the contact stresses beneath a perfectly flexible plate will be examined in relation to Figs. 2–4.

In the latter instance, we note from (14), (16) and (20) that as $R_A \to \infty$, the contact stress at the interface reduces to

$$\sigma_{zz}(r, 0) = \frac{p_0 a}{2\sqrt{(a^2 - r^2)}}. \tag{28}$$

This result is in agreement with the analytical results derived by Elliott[1] and Sveklo[4] for the contact stress distribution beneath a rigid circular flat punch resting in smooth contact with a transversely isotropic elastic halfspace. Here, the contact stress is uninfluenced by the degree of transverse isotropy of the halfspace region.

For circular plates of intermediate relative rigidity, the contact stresses depend on the flexibility characteristics of the circular plate and the material characteristics of the transversely isotropic elastic medium. The energy estimate of the contact stress distribution at the frictionless interface can be written in the form

$$\bar{\sigma}_{zz} = \frac{\sigma_{zz}(r, 0)}{p_0} = \frac{1}{\sqrt{(1-(r/a)^2)}}\left\{c_0^* + c_2^*\left[\sum_{i=0}^{3} \eta_{2i}\left(\frac{r}{a}\right)^{2i}\right]\right\} \tag{29}$$

where the constants c_i^* depend on the relative rigidity parameter R_A. This latter parameter can be rewritten as

$$R_A = KR^* \tag{30}$$

where K is a reduced relative rigidity parameter defined by

$$K = \frac{\pi}{12}\frac{E_b}{c_{44}(1-\nu_b^2)}\left(\frac{h}{a}\right)^3 \tag{31}$$

and

$$R^* = \frac{(k_1 - k_2)\sqrt{(\nu_1\nu_2)}}{\frac{c_{33}}{c_{44}}\{k_1\sqrt{(\nu_2)}(1+k_2) - k_2\sqrt{(\nu_1)}(1+k_1)\} - \frac{c_{13}}{c_{44}}\{\nu_1\sqrt{(\nu_2)}(1+k_2) - \nu_2\sqrt{(\nu_1)}(1+k_1)\}}. \tag{32}$$

The influence of the degree of transverse isotropy and the relative rigidity on the contact stress at the interface is examined by carrying out numerical computations for certain specific materials which display transversely isotropic properties. The material constants c_{ij}, k_i and ν_i characterizing transversely isotropic materials such as magnesium and cadmium are reported by Chen[12], Atsumi and Itou[13] and Dahan and Zarka[14]. These properties together with material constants corresponding approximately to an isotropic material are listed in Tables 1 and 2. The contact stress distributions derived for these three categories of materials are

Table 1. Elastic constants c_{ij} used (in units of 10^{11} dyn/cm^2)

C_{ij}	c_{44}	c_{11}	c_{33}	c_{12}	c_{13}
Approximate isotropy	0.99997	3.5	3.5	1.5	1.5
Magnesium	1.64	5.97	6.17	2.62	2.17
Cadmium	1.56	11.00	4.69	4.04	3.83

Table 2. Values of ν_i and k_i ($i = 1, 2$)

	ν_1	ν_2	k_1	k_2
Approximate isotropy	1.00930	0.99078	1.01305	0.98712
Magnesium	2.05017	0.50411	2.78203	0.35945
Cadmium	1.04862	0.40660	1.85062	0.54036

illustrated in Figs. 2–4. Values assigned for the reduced relative rigidity parameter K range from 10^6 to 10^{-4}. The relative rigidity of 10^6 corresponds, approximately, to a rigid circular plate. The contact stress distributions given in Figs. 2–4 indicate that at this upper limit of K the contact stresses are uninfluenced by the degree of transverse isotropy and the results corresponds accurately to the contact stress distribution beneath the rigid circular punch. Similarly at the lower limit of K ($= 10^{-4}$) the contact stresses appear to uninfluenced by the degree of transverse isotropy of the elastic medium. This lower limit therefore corresponds to the energy solution for the perfectly flexible plate. Theoretically, this contact stress distribution should be uniform. The energy solution, however, gives a non-uniform result indicating that the contact stress distribution at this range of relative rigidity is sensitive to the prescribed deflected shape. To accurately depict the contact stress distribution applicable for the case $K \to 0$ it is necessary to include further terms in the expansion for $w(r)$ defined by (13). For the isotropic case (Fig. 2) the contact stresses compare favourably with equivalent results given by Brown[15] who employs a power series expansion technique for the analysis of the interaction problem. Similar results are given by Borowicka[16]. Admittedly, this paper examines only a few specialized cases of transverse isotropy. As such no general conclusions can be made with regard to the nature of the contact stress distribution at the frictionless interface. The results given here, however, indicate that the development of tensile contact stresses can be suppressed in unilateral contact problems involving circular plates which are uniformly loaded over its entire surface area.

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REFERENCES

1. H. A. Elliott, *Proc. Camb. Phil. Soc.* **45**, 621–630 (1949).
2. R. T. Shield, *Proc. Camb. Phil. Soc.* **47**, 401–409 (1951).
3. A. H. England, *Proc. Camb. Phil. Soc.* **58**, 539–547 (1962).
4. V. A. Sveklo, *J. Appl. Math. Mech.* **34**, 165–171 (1970).
5. S. P. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells.* McGraw-Hill, New York (1959).
6. I. S. Sokolnikoff, *Mathematical Theory of Elasticity.* McGraw-Hill, New York (1956).
7. Y. Weitsman, *J. Appl. Mech.* **36**, 198–202 (1969).
8. G. M. L. Gladwell and K. R. P. Iyer, *J. Elasticity* **4**, 115–130 (1974).
9. G. M. L. Gladwell, *J. Appl. Mech.* **43**, 263–267 (1976).
10. A. D. de Pater and J. J. Kalker (Eds), The mechanics of contact between deformable bodies. *Proc. I.U.T.A.M. Symp., Enschede.* Delft Univ. Press (1974).
11. A. P. S. Selvadurai, *Elastic Analysis of Soil-Foundation Interaction: Developments in Geotechnical Engineering,* Vol. 17. Elsevier, Amsterdam (1979).
12. W. T. Chen, *J. Appl. Mech.* **33**, 347–355 (1966).
13. A. Atsumi and S. Itou, *J. Appl. Mech.* **41**, 507–511 (1974).
14. M. Dahan and J. Zarka, *Int. J. Solids Structures* **13**, 229–238 (1977).
15. P. T. Brown, *Geotechnique* **19**, 399–404 (1969).
16. H. Borowicka, *Ing. Arch.* **10**, 113–125 (1939).
17. A. E. Green and W. Zerna, *Theoretical Elasticity.* Clarendon Press, Oxford (1968).

APPENDIX A

The stress-strain relationships for the transversely isotropic material is given by

$$\sigma_{xx} = c_{11}\epsilon_{xx} + c_{12}\epsilon_{yy} + c_{13}\epsilon_{zz}; \qquad \sigma_{yz} = c_{44}\epsilon_{yz}$$

$$\sigma_{yy} = c_{12}\epsilon_{xx} + c_{11}\epsilon_{yy} + c_{13}\epsilon_{zz}; \qquad \sigma_{zx} = c_{44}\epsilon_{zx}$$

$$\sigma_{zz} = c_{13}(\epsilon_{xx} + \epsilon_{yy}) + c_{33}\epsilon_{zz}; \qquad \sigma_{xy} = \frac{1}{2}(c_{11} - c_{12})\epsilon_{xy}$$

where σ and ϵ are the stress and strain tensors referred to the rect. cartesian system.

Following Green and Zerna[17] we introduce a set of non-dimensional parameters ν_α and k_α ($\alpha = 1, 2$) which are dependent on the five elastic constants c_{ij}. The pair ν_1 and ν_2 are roots of the equation

$$c_{11}c_{44}\nu^2 + [c_{13}(2c_{44} + c_{13}) - c_{11}c_{33}]\nu + c_{33}c_{44} = 0$$

and k_i ($i = 1, 2$) are defined by

$$k_i = \frac{c_{11}\nu_i - c_{44}}{c_{13} + c_{44}}, \qquad (i = 1, 2).$$

Also γ_i ($i = 1, 2$) are given by

$$\gamma_i = \frac{1}{c_{44}}\left\{\frac{c_{33}k_i - c_{13}\nu_i}{(1 + k_i)\sqrt{(\nu_i)}}\right\}.$$

The parameters η_{2i} ($i = 0, 1, 2, 3$) and χ_i ($i = 1, 2, 3, 4$) are given by

$$\eta_0 = -\left(2 + \frac{8\lambda_1}{9} + \frac{16}{25}\lambda_2\right); \qquad \eta_2 = 4 - \frac{32}{9}\lambda_1 - \frac{32}{25}\lambda_2$$

$$\eta_4 = \frac{64}{9}\lambda_1 - \frac{128}{25}\lambda_2; \qquad \eta_6 = \frac{256}{25}\lambda_2$$

and

$$\chi_1 = \eta_0 + \frac{2}{3}(1 + \eta_2) + \frac{8}{15}(\lambda_1 + \eta_4) + \frac{16}{35}(\lambda_2 + \eta_6)$$

$$\chi_2 = \frac{2}{3}\eta_0 + \frac{8}{15}(\lambda_1\eta_0 + \eta_2) + \frac{16}{35}(\lambda_2\eta_0 + \lambda_1\eta_2 + \eta_4)$$
$$+ \frac{128}{315}(\eta_2\lambda_2 + \eta_4\lambda_1 + \eta_6) + \frac{256}{693}(\eta_4\lambda_2 + \eta_6\lambda_1) + \frac{1024}{3003}\eta_6\lambda_2$$

$$\chi_3 = 8 + 32\lambda_1 + \frac{144}{3}\lambda_2 + 144\lambda_1\lambda_2 + \frac{128}{3}\lambda_1^2 + \frac{1188}{5}\lambda_2^2$$
$$+ -(1 - \nu_b)\{4 + 16\lambda_1 + 24\lambda_2 + 48\lambda_1\lambda_2 + 16\lambda_1^2 + 36\lambda_2^2\}$$

$$\chi_4 = \frac{1}{2} + \frac{\lambda_1}{3} + \frac{\lambda_2}{4}.$$

The constants m_0 and m_2 are defined by

$$m_0 = \frac{c_0^*}{3} + c_2^*\left\{\frac{\eta_0}{3} + \frac{2}{15}\eta_2 + \frac{8}{105}\eta_4 + \frac{16}{315}\eta_6\right\}$$

$$m_2 = \left\{c_0^* + c_2^*\left(\eta_0 + \frac{2}{3}\eta_2 + \frac{8}{15}\eta_4 + \frac{36}{105}\eta_6\right)\right\}\ln 2$$
$$+ -\left\{c_0^* + c_2^*\left(\eta_0 + \frac{5}{9}\eta_2 + \frac{94}{225}\eta_4 + \frac{1276}{3695}\eta_6\right)\right\}$$

where

$$\{c_0^*; c_2^*\} = \{(\chi_1\chi_4 - 2\chi_2 - R_A\chi_3); (\chi_1 - 2\chi_4)\}[\chi_1^2 - 4\chi_2 - 2R_A\chi_3]^{-1}.$$